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The Degree of Local Approximation of Functions in $C_1[0, 1]$ by Bernstein Polynomials

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1. INTRODUCTION AND SUMMARY

For functions $f \in C[0, 1]$ the expression

$$B_n(f; x) := \sum_{k=0}^{n} f(k/n)\, p_{n,k}(x), \tag{1.1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad (x \in [0, 1];\, n = 1, 2,...;\, k = 0, 1,..., n)$$

is called the Bernstein polynomial of order n of f. Popoviciu [6] proved that for all $n \in \mathbb{N}$ and all $f \in C[0, 1]$,

$$\max_{0 \leqslant x \leqslant 1} | B_n(f; x) - f(x)| \leqslant A\omega(f; n^{-1/2}) \tag{1.2}$$

with $A = \frac{3}{2}$. Here $\omega(f; \delta)$ denotes the modulus of continuity of f, i.e.,

$$\omega(f; \delta) = \sup_{|x-y| \leqslant \delta} | f(x) - f(y)| \qquad (\delta > 0).$$

The best constant possible in (1.2) was obtained by Sikkema [9, 10], viz.,

$$A^* = (4306 + 837(6)^{1/2})/5832 = 1.089887.[1]$$

Esseen [2] showed that the smallest B such that for all $f \in C[0, 1]$

$$\limsup_{n\to\infty} \max_{0 \leqslant x \leqslant 1} \frac{| B_n(f; x) - f(x)|}{\omega(f; n^{-1/2})} \leqslant B$$

[1] Here and elsewhere numbers are rounded to the last digit shown.

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is given by

$$B = 2 \sum_{j=0}^{\infty} (j+1)\{\Phi(2j+2) - \Phi(2j)\} = 1.045564,$$

where

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-\tfrac{1}{2}t^2)\, dt.$$

The purpose of this paper is to derive results analogous to those of Sikkema and Esseen for functions in C_1 (for notational reasons we prefer to write C_1 rather than $C_1[0, 1]$; also we shall often consider functions defined on $(-\infty, \infty)$). More precisely, let $\omega_1(f; \delta) := \omega(f'; \delta)$ and let

$$c_n := \sup_{f\in C_1} \max_{0\leqslant x\leqslant 1} \frac{n^{1/2} \mid B_n(f; x) - f(x)\mid}{\omega_1(f; n^{-1/2})}; \tag{1.3}$$

then we shall obtain

$$c^{(j)} := \sup_{n\geqslant j} c_n \qquad (j = 1, 2) \tag{1.4}$$

and

$$c := \lim_{n\to\infty} c_n . \tag{1.5}$$

A first result in this direction is due to Lorentz [5, p. 21], who proves that $c^{(1)} \leqslant \frac{3}{4}$.

Section 2 contains two preliminary lemmas. In order to obtain *local* results, i.e., results still containing x, in Section 3 we introduce the *extremal functions* $\tilde{f}_n$, containing x as a parameter, satisfying

$$c_n(x) := \sup_{f\in C_1} \frac{n^{1/2} \mid B_n(f; x) - f(x)\mid}{\omega_1(f; n^{-1/2})} = n^{1/2}\{B_n(\tilde{f}_n ; x) - \tilde{f}_n(x)\}, \tag{1.6}$$

where $c_n(x)$ and $c(x) := \lim_{n\to\infty} c_n(x)$ measure the degree of local approximation. From (1.3) and (1.6), together with the fact that $c_n(x)$ turns out to be continuous (cf. (4.1)), it follows that

$$c_n = \max_{0\leqslant x\leqslant 1} c_n(x). \tag{1.7}$$

In Section 4 we calculate $c_5(x)$ and c_5, in Section 5 it is proved that $c^{(1)} = c_1 = \frac{1}{4}$ and in Section 6 that $c^{(2)} = c_5 = (2(5)^{1/2} - 1)/16 = 0.217008$. Finally, in Section 7 we obtain $\lim_{n\to\infty} c_n(x)$ and $\lim_{n\to\infty} c_n$.

Remark 1.1. As *linear functions* are left intact by the Bernstein operators, they are of no interest to our problems. Furthermore, expressions such as those in the right-hand side of (1.3) are undefined for linear functions; therefore we shall disregard them, without indicating this in our notation.

Proofs in this paper have been kept rather brief; for full details we refer to [7].

2. Preliminary Results

LEMMA 2.1. *Let*

$$T_{n,s}(x) := \sum_{k=0}^{n} (k - nx)^s p_{n,k}(x) \qquad (s = 0, 1, 2, \ldots)$$

and $X := x(1-x)$; *then*

$$T_{n,0}(x) = 1, \qquad T_{n,1}(x) = 0, \qquad T_{n,2}(x) = nX, \tag{2.1}$$

$$T_{n,6}(x) = 15n^3X^3 + 5n^2X^2(5 - 26X) + nX(1 - 30X + 120X^2). \tag{2.2}$$

Proof. Recursion relations for the $T_{n,s}$ can be found in [5, p. 14].

LEMMA 2.2. *For* $S_n(x)$ *defined by*

$$S_n(x) = \tfrac{1}{2}n^{1/2} \sum_{k=0}^{n} |(k/n) - x|\, p_{n,k}(x) \tag{2.3}$$

one has, $[a]$ *denoting the largest integer not exceeding* a,

$$S_n(x) = n^{-1/2}(n - m)\binom{n}{m} x^{m+1}(1 - x)^{n-m} \qquad (m = [nx]); \tag{2.4}$$

$S_n(x)$ *has a unique maximum* $S_{n,m}$ *on* $[m/n, (m+1)/n]$ *at* $(m+1)/(n+1)$ *for* $m = 0, 1, \ldots, [(n-1)/2] =: m^*$, *and*

$$\begin{aligned} \| S_n \| &:= \max_{0 \leqslant x \leqslant 1} S_n(x) = \max_{0 \leqslant x \leqslant 1/2} S_n(x) \\ &= S_{n,m^*} > S_{n,m^*-1} > \cdots > S_{n,1} > S_{n,0}\,; \end{aligned} \tag{2.5}$$

$$\begin{aligned} \tfrac{1}{4} &= \| S_1 \| > \| S_3 \| > \| S_5 \| > \cdots, \\ (4/27)\, 2^{1/2} &= \| S_2 \| > \| S_4 \| > \| S_6 \| > \cdots. \end{aligned} \tag{2.6}$$

Proof. Equation (2.4) can be proved by using Hilfssatz 1 of [9]. It is obvious from (2.4) that $S_n(x)$ has a unique maximum on $[m/n, (m+1)/n]$ at $(m+1)/(n+1)$. For the proofs of (2.5) and (2.6), which are straightforward but somewhat tedious, we refer to [7].

The numerical values of $\| S_n \|$ for $n = 1, 2, \ldots, 30$ are shown in Table I of Section 6.

3. The Extremal Functions

In this section we construct the functions $\tilde{f}_n$ satisfying (1.6). First, replacing $n^{-1/2}$ by δ, we construct extremal functions $\tilde{f}$ in the slightly more general setting, where errors are measured in terms of $\omega_1(f; \delta)$ rather than $\omega_1(f; n^{-1/2})$. Abbreviating

$$\Delta_n(f; x) := B_n(f; x) - f(x), \tag{3.1}$$

by (1.1) we have

$$\Delta_n(f; x) = \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} f'(t)\, dt. \tag{3.2}$$

We prove the following theorem.

THEOREM 3.1. *For each $n \in \mathbb{N}$, for each $x_0 \in [0, 1]$ and each $\delta > 0$,*

$$\sup_{f \in C_1} \frac{|\Delta_n(f; x_0)|}{\omega_1(f; \delta)} = \Delta_n(\tilde{f}; \delta), \tag{3.3}$$

where $\tilde{f}$, which depends on x_0 and δ, is defined (for all real x) by

$$\begin{aligned} &\tilde{f}(x_0) = 0, \\ &\tilde{f}'(x) = j + \tfrac{1}{2} \qquad (j\delta < x - x_0 \leqslant (j+1)\delta; j = 0, \pm 1, \pm 2, \ldots). \end{aligned} \tag{3.4}$$

The functions $\tilde{f}$ will be called extremal. We shall prove Theorem 3.1 in a number of small steps, stated as lemmas, which gradually narrow the class of functions to be considered. We first replace class C_1 by the slightly wider class K_δ defined as follows:

$$K_\delta = \{f \in C; f' \text{ is continuous with the exception of finitely many jumps in finite intervals, } 0 < \omega_1(f; \delta) \leqslant 1\}, \tag{3.5}$$

where C denotes the set of continuous functions. Here $\omega_1 > 0$ excludes the linear functions (cf. Remark 1.1), and $\omega_1 \leqslant 1$ is a simple matter of scale. In order to avoid needless difficulties at the boundary points 0 and 1, here and elsewhere we continue all functions to $(-\infty, \infty)$ in such a way that their essential properties, e.g., convexity, are preserved. We now state and prove our lemmas.

LEMMA 3.1.

$$\sup_{f \in C_1} \frac{|\Delta_n(f; x_0)|}{\omega_1(f; \delta)} = \sup_{f \in K_\delta} \frac{|\Delta_n(f; x_0)|}{\omega_1(f; \delta)} \qquad (x_0 \in [0, 1]).$$

Proof. On [0, 1] $f \in K_\delta$ is the pointwise limit of functions in C_1 with the same value of $\omega_1(f; \delta)$, as is easily seen by approximating f' by functions in C and integrating. The lemma then follows from the continuity of B_n with respect to pointwise convergence.

LEMMA 3.2.

$$\sup_{f\in K_\delta} \frac{|\Delta_n(f; x_0)|}{\omega_1(f; \delta)} = \sup_{\substack{f\in K_\delta \\ f \text{ convex}}} \frac{\Delta_n(f; x_0)}{\omega_1(f; \delta)} \qquad (x_0 \in [0, 1]).$$

Proof. As f may be replaced by $-f$, without loss of generality we take $f \in K_\delta$ such that $\Delta_n(f; x_0) \geqslant 0$. From f we construct a convex function $\check{f}$ as follows. Take $\check{f}(x_0) = f(x_0)$ and define $\check{f}'$ by

$$\begin{aligned} \check{f}'(x) &= \inf_{x\leqslant u\leqslant x_0} f'(u) \qquad (x \leqslant x_0) \\ &= \sup_{x_0\leqslant u\leqslant x} f'(u) \qquad (x \geqslant x_0). \end{aligned} \tag{3.6}$$

Clearly, $\check{f}'$ is nondecreasing, i.e., $\check{f}$ is convex. We now prove that $\omega_1(\check{f}; \delta) \leqslant \omega_1(f; \delta)$. If on $[x, x+\delta]$ the function $\check{f}'$ varies by ϵ, i.e., if $\check{f}'(x+\delta) - \check{f}'(x) = \epsilon$, then by the definition of $\check{f}'$, for each $\eta > 0$ there are x_1 and x_2 with $x \leqslant x_1 < x_2 \leqslant x + \delta$ and such that $f'(x_2) - f'(x_1) \geqslant \epsilon - \eta$. This implies that $\omega_1(\check{f}; \delta) \leqslant \omega_1(f; \delta) \leqslant 1$. The remaining conditions for $\check{f}$ to be in K_δ are easily checked. Finally, as $\check{f}' \leqslant f'$ for $x \leqslant x_0$ and $\check{f}' \geqslant f'$ for $x \geqslant x_0$, it follows from (3.2) that $\Delta_n(\check{f}; x_0) \geqslant \Delta_n(f; x_0)$ and the lemma is proved.

For fixed x_0 and arbitrary f on $(-\infty, \infty)$ we now define a continuous function f^* by

$$\left.\begin{aligned} &f^*(x_0 + j\delta) = f(x_0 + j\delta) \\ &f^* \text{ is linear on } (x_0 + j\delta, x_0 + j\delta + \delta) \end{aligned}\right. \qquad (j = 0, \pm 1, \pm 2,\ldots). \tag{3.7}$$

LEMMA 3.3. *Let f be convex and $f \in K_\delta$, then f^* is convex and $f^* \in K_\delta$.*

Proof. That f^* is convex is trivial. To prove that $f^* \in K_\delta$, we show that $\omega_1(f^*; \delta) \leqslant \omega_1(f; \delta) \leqslant 1$; the other conditions for K_δ are easily seen to hold. We proceed as follows. If t is not of the form $x_0 + j\delta$ then $f^{*\prime}(t)$ is well defined. If $t = x_0 + j\delta$, we define $f^{*\prime}(t)$ by continuity from the left. Now, for any two points t_1 and t_2 with $t_1 < t_2 \leqslant t_1 + \delta$ we have for some integer j

$$\begin{aligned} 0 &\leqslant f^{*\prime}(t_2) - f^{*\prime}(t_1) \leqslant f^{*\prime}(t_1 + \delta) - f^{*\prime}(t_1) \\ &= \frac{f(x_0 + j\delta + \delta) - f(x_0 + j\delta)}{\delta} - \frac{f(x_0 + j\delta) - f(x_0 + j\delta - \delta)}{\delta} \\ &= (1/\delta) \int_{j\delta}^{j\delta+\delta} f'(x_0 + t)\, dt - (1/\delta) \int_{j\delta-\delta}^{j\delta} f'(x_0 + t)\, dt \\ &= (1/\delta) \int_{j\delta}^{j\delta+\delta} \{f'(x_0 + t) - f'(x_0 + t - \delta)\}\, dt, \end{aligned}$$

from which it follows that $\omega_1(f^*; \delta) \leqslant \omega_1(f; \delta)$.

LEMMA 3.4. *Let f be convex and $f \in K_\delta$, then*

$$\frac{\Delta_n(f^*; x_0)}{\omega_1(f^*; \delta)} \geqslant \frac{\Delta_n(f; x_0)}{\omega_1(f; \delta)} \qquad (x_0 \in [0, 1]).$$

Proof. As $f^* \geqslant f$ on $[0, 1]$, by the positivity of the operator B_n we have $B_n(f^*; x) \geqslant B_n(f; x)$ for all $x \in [0, 1]$. As $f^*(x_0) = f(x_0)$ by definition, and $\omega_1(f^*; \delta) \leqslant \omega_1(f; \delta)$ by the proof of Lemma 3.3, the lemma follows.

We now define a class $K_\delta{}^*$ of piecewise linear functions by

$$K_\delta{}^* = \{f \in K_\delta\,; f \text{ convex}, f^* = f, f(x_0) = 0, f'(x) = \tfrac{1}{2} \text{ for } x_0 < x \leqslant x_0 + \delta\},$$

where the restrictions on $f(x_0)$ and f' are not essential, as $B_n(l; x) \equiv l(x)$ for every linear function l. From the preceding four lemmas we now obtain

LEMMA 3.5.

$$\sup_{f \in C_1} \frac{|\Delta_n(f; x_0)|}{\omega_1(f; \delta)} = \sup_{f \in K_\delta{}^*} \frac{\Delta_n(f; x_0)}{\omega_1(f; \delta)} \qquad (x_0 \in [0, 1]).$$

We are now ready to prove the main result of this section.

Proof of Theorem 3.1. For $f \in K_\delta{}^*$ we have in view of (3.2)

$$\frac{\Delta_n(f; x_0)}{\omega_1(f; \delta)} = \sum_{k=0}^{n} p_{n,k}(x_0) \int_{x_0}^{k/n} \frac{f'(t)}{\omega_1(f; \delta)}\, dt, \tag{3.8}$$

where f'/ω_1 is a nondecreasing stepfunction with largest step equal to 1, i.e., with modulus of continuity equal to 1. It is obvious from (3.8) that Δ_n/ω_1 is maximal if all steps of f'/ω_1 are equal to 1, i.e., if $f'/\omega_1 = \tilde{f}'$ as defined in (3.4). This proves the theorem.

We conclude this section by giving explicit expressions for $\tilde{f}$ and $\Delta_n(\tilde{f}; x_0)$. From (3.4) we get by integration

$$\tilde{f}(x) = \tfrac{1}{2}\, |x - x_0| + \sum_{j=1}^{\infty} (|x - x_0| - j\delta)_+, \tag{3.9}$$

where $a_+ := \max(a, 0)$. As $\tilde{f}(x_0) = 0$ we have $\Delta_n(\tilde{f}; x_0) = B_n(\tilde{f}; x_0)$, and hence

$$\Delta_n(\tilde{f}; x_0) = \frac{1}{2} \sum_{k=0}^{n} \left| \frac{k}{n} - x_0 \right| p_{n,k}(x_0)$$
$$+ \sum_{j=1}^{\infty} \sum_{|(k/n) - x_0| \geqslant j\delta} \left(\left| \frac{k}{n} - x_0 \right| - j\delta \right) p_{n,k}(x_0). \tag{3.10}$$

The extremal functions with $\delta = n^{-1}$ have been used in [8] to obtain the solution of similar problems as described in Section 1, with $\omega_1(f; n^{-1/2})$ replaced by $\omega_1(f; n^{-1})$. If $\delta = n^{-1/2}$, we write $\tilde{f}_n$ instead of $\tilde{f}$ (cf. (1.6)).

4. Calculation of $c_n(x)$ and c_n for Small n

In this section we explicitly calculate $c_5(x)$ and c_5. The calculation of $c_5(x)$ also serves as an example of the difficulties involved, and the values of c_1, c_2, c_3, and c_4 are given without any computations. For $n > 5$ the amount of work of this method rapidly becomes prohibitive.

To calculate $c_n(x)$ we use (cf. (1.6), Theorem 3.1, and (3.10) with $\delta = n^{-1/2}$)

$$c_n(x) = n^{1/2}\Delta_n(\tilde{f}_n\,; x) = \frac{1}{2}n^{1/2}\sum_{k=0}^{n}\left|\frac{k}{n} - x\right| p_{n,k}(x)$$
$$+ n^{1/2}\sum_{j=1}^{\infty}\sum_{|(k/n)-x|\geqslant jn^{-1/2}}\left(\left|\frac{k}{n} - x\right| - jn^{-1/2}\right)p_{n,k}(x). \quad (4.1)$$

Taking $n = 5$ we get seven different expressions for $c_5(x)$, which we denote by $c_{5,1}(x),\dots, c_{5,7}(x)$.

$$c_{5,1}(x) \equiv 5^{1/2}x(1-x)^5 + 5^{1/2}\left\{\left(1 - x - \frac{1}{5^{1/2}}\right)x^5 + (4 - 5x - 5^{1/2})\,x^4(1-x)\right.$$
$$\left. + (6 - 10x - 2(5)^{1/2})\,x^3(1-x)^2 + \left(1 - x - \frac{2}{5^{1/2}}\right)x^5\right\}$$
$$\text{for } x \in \left[0, 1 - \frac{2}{5^{1/2}}\right] =: J_1,$$

$$c_{5,2}(x) \equiv c_{5,1}(x) - 5^{1/2}\left(1 - x - \frac{2}{5^{1/2}}\right)x^5 \quad \text{for } x \in \left[1 - \frac{2}{5^{1/2}}, \frac{3}{5} - \frac{1}{5^{1/2}}\right] =: J_2,$$

$$c_{5,3}(x) \equiv c_{5,2}(x) - 5^{1/2}(6 - 10x - 2(5)^{1/2})\,x^3(1-x)^2$$
$$\text{for } x \in \left[\frac{3}{5} - \frac{1}{5^{1/2}}, \frac{1}{5}\right] =: J_3,$$

$$c_{5,4}(x) \equiv 4(5)^{1/2}\,x^2(1-x)^4 + 5^{1/2}\left\{\left(1 - x - \frac{1}{5^{1/2}}\right)x^5\right.$$
$$\left. + (4 - 5x - 5^{1/2})\,x^4(1-x)\right\} \quad \text{for } x \in \left[\frac{1}{5}, \frac{4}{5} - \frac{1}{5^{1/2}}\right] =: J_4,$$

$$c_{5,5}(x) \equiv c_{5,4}(x) - 5^{1/2}(4 - 5x - 5^{1/2})\,x^4(1-x)$$
$$\text{for } x \in \left[\frac{4}{5} - \frac{1}{5^{1/2}}, \frac{2}{5}\right] =: J_5,$$

$$c_{5,6}(x) \equiv 6(5)^{1/2}\, x^3(1-x)^3 + 5^{1/2}\left(1 - x - \frac{1}{5^{1/2}}\right) x^5 \qquad \text{for } x \in \left[\frac{2}{5}, \frac{1}{5^{1/2}}\right] =: J_6,$$

$$c_{5,7}(x) \equiv c_{5,6}(x) + 5^{1/2}\left(x - \frac{1}{5^{1/2}}\right)(1-x)^5 \qquad \text{for } x \in \left[\frac{1}{5^{1/2}}, \frac{1}{2}\right] =: J_7.$$

It is quite elementary to show that

$$\max_{x \in J_7} c_5(x) = c_5(\tfrac{1}{2}) = (2(5)^{1/2} - 1)/16 = 0.217008.$$

To prove that, in fact, $c_5 = c_5(\frac{1}{2})$, we compare $c_5(x)$ on $J_1, \ldots, J_6$ with this number. By straightforward calculation one shows that

$$c_5(x) < 0.1368 \text{ on } J_1, \qquad c_5(x) < 0.1542 \text{ on } J_2, \qquad c_5(x) < 0.1558 \text{ on } J_3,$$
$$c_5(x) < 0.2011 \text{ on } J_4, \qquad c_5(x) < 0.1989 \text{ on } J_5, \qquad c_5(x) = 0.2069 \text{ on } J_6,$$

and hence that $c_5 = (2(5)^{1/2} - 1)/16$.

The calculation of c_1, c_2, c_3, and c_4 is similar to that of c_5, but simpler. We state their values in the following theorem. For more details we refer to [7].

THEOREM 4.1. *For the c_n as defined in* (1.3) (*see also* (1.7)) *one has*

$$\begin{aligned}
c_1 &= c_1(1/2) = 1/4 = 0.250000,\\
c_2 &= c_2(1/3) = (4/27)(2)^{1/2} = 0.209513,\\
c_3 &= c_3(1/2) = (1/8)(3)^{1/2} = 0.216506,\\
c_4 &= c_4(2/5) = 664/3125 = 0.212480,\\
c_5 &= c_5(1/2) = (2(5)^{1/2} - 1)/16 = 0.217008.
\end{aligned}$$

5. A Simple Proof of $c^{(1)} = \frac{1}{4}$

From formula (3.9) with $\delta = n^{-1/2}$ we have

$$\tilde{f}_n(x) = \tfrac{1}{2}\,|x - x_0| + \sum_{j=1}^{\infty} (|x - x_0| - jn^{-1/2})_+ . \tag{5.1}$$

We compare $\tilde{f}_n$ with a quadratic function q_n defined by

$$q_n(x) = \frac{1}{4}\left(\frac{x_0(1-x_0)}{n}\right)^{1/2} + \frac{1}{4}\left(\frac{n}{x_0(1-x_0)}\right)^{1/2} (x - x_0)^2. \tag{5.2}$$

The function q_n is easily seen (cf. (2.1)) to have the following properties:

$$q_n(x) \geqslant \tilde{f}_n(x) \qquad \text{for all } x, \tag{5.3}$$

$$B_n(q_n\,;x_0) = \tfrac{1}{2}(x_0(1-x_0)/n)^{1/2}. \tag{5.4}$$

Now, using (4.1), (3.1), and the fact that $\tilde{f}_n(x_0) = 0$, by the positivity of B_n we obtain from (5.3 and (5.4)

$$c_n(x_0) = n^{1/2}B_n(\tilde{f}_n\,;x_0) \leqslant n^{1/2}B_n(q_n\,;x_0) = \tfrac{1}{2}(x_0(1-x_0))^{1/2} \leqslant \tfrac{1}{4}. \tag{5.5}$$

From (5.5), together with the fact that $c_1 = c_1(\frac{1}{2}) = \frac{1}{4}$ (cf. Theorem 4.1), we obtain one of the main results of this paper, viz.,

THEOREM 5.1.

$$c^{(1)} := \sup_{n\geqslant 1}\sup_{f\in C_1}\max_{0\leqslant x\leqslant 1}\frac{n^{1/2}\,|\,B_n(f;x)-f(x)|}{\omega_1(f;n^{-1/2})} = \frac{1}{4}.$$

Remark 5.1. Similarly, by comparing $\tilde{f}_n$ with a quadratic function $\hat{q}_n$ such that $\hat{q}_n \leqslant \tilde{f}_n$, we obtain a lower bound for $c_n(x)$. Combining this result with (5.5), we get

$$\tfrac{1}{2}x(1-x) \leqslant c_n(x) \leqslant \tfrac{1}{2}(x(1-x))^{1/2}. \tag{5.6}$$

6. DETERMINATION OF $c^{(2)}$

The bound $c^{(1)}$ is unsatisfactory for the following reasons. It is attained for $n = 1$, which seems a bit too special, and the value of $c^{(1)}$ differs rather much from both the next few values of c_n (cf. Theorem 4.1) and the limiting value c (cf. Theorem 7.1). We are therefore led to look for $c^{(2)} = \sup_{n\geqslant 2} c_n$. The main result of this section is

THEOREM 6.1.

$$c^{(2)} := \sup_{n\geqslant 2}\sup_{f\in C_1}\max_{0\leqslant x\leqslant 1}\frac{n^{1/2}\,|\,B_n(f;x)-f(x)|}{\omega_1(f;n^{-1/2})}$$

$$= c_5 = \frac{2(5)^{1/2}-1}{16} = 0.217008497.$$

Proof. We start from (1.7) and (4.1), and we write for fixed $x_0 \in [0, 1]$ (cf. (2.3))

$$c_n(x_0) = S_n(x_0) + R_n(x_0),$$

where R_n is defined by

$$R_n(x_0) = n^{1/2}B_n(Q_n\,;\,x_0)$$

with

$$Q_n(x) := \sum_{j=1}^{\infty} (|\,x - x_0\,| - jn^{-1/2})_+\,.$$

We give a bound for $R_n(x_0)$ by estimating $Q_n(x)$ by a polynomial $P_n(x)$ defined as

$$P_n(x) = (5^5/6^6)\,n^{5/2}(x - x_0)^6.$$

It is easily verified that $Q_n(x) \leqslant P_n(x)$ for all x, and hence, by the positivity of B_n, that

$$R_n(x_0) \leqslant n^{1/2}B_n(P_n\,;\,x_0) = (5^5/6^6)\,n^{-3}T_{n,6}(x_0), \tag{6.1}$$

with $T_{n,6}$ as given in (2.2). As $T_{n,6}(x)$ is maximal at $x = \frac{1}{2}$ for all $n \geqslant 4$, it follows that

$$R_n(x_0) \leqslant R_n{}^* := \frac{5^5}{6^6}\,n^{-3}T_{n,6}\left(\frac{1}{2}\right)$$

$$= \frac{5^6}{2^{12}3^5}\left(1 - \frac{2}{n} + \frac{16}{15n^2}\right) = 0.015699\left(1 - \frac{2}{n} + \frac{16}{15n^2}\right). \tag{6.2}$$

Theorem 4.1 takes care of the cases $n = 2, 3, 4$. Hence, it is sufficient to prove that for $n \geqslant 6$, and all $x \in [0, 1]$

$$S_n(x) \leqslant 0.217008 - 0.015699(1 - (2/n) + 16/(15n^2)). \tag{6.3}$$

In Table I the values of $\|\,S_n\,\|$ ($n = 1, 2,..., 30$) (cf. Lemma 2.2) and of $\alpha_n := 0.217008 - R_n{}^*$ ($n = 4, 5,..., 30$) are given, and from this table it follows that (6.3) holds for all these values of n with the exception of 7, 9, and 11. As the values of $n > 30$ are taken care of by the monotonicity of $\|\,S_{2j}\,\|$ and $\|\,S_{2j+1}\,\|$ (cf. (2.6)), only the cases 7, 9, and 11 remain. We treat these cases separately and briefly; for details we again refer to [7].

Case $n = 7$. It can easily be shown that on $[0, 0.48]$ one has $S_7(x) \leqslant S_7(0.48) = 0.205380$. As $R_7{}^* = 0.011555$, it follows that $c_7(x) \leqslant 0.216935$. Therefore we may restrict x to $[0.48, 0.50]$. On this interval we have (cf. (4.1))

$$c_7(x) = 20(7)^{1/2}\,x^4(1 - x)^4 + 7^{1/2}\{(1 - x)^7(x - 7^{-1/2}) + x^7(1 - x - 7^{-1/2})\},$$

which is maximal at $x = \frac{1}{2}$, with $c_7(\frac{1}{2}) = (11(7)^{1/2} - 2)/128 = 0.211744$. It follows that $c_7 < c_5$.

TABLE I

n	$\|S_n\|$	α_n	n	$\|S_n\|$	α_n
1	0.250000		16	0.202246	0.203207
2	0.209513		17	0.202425	0.203099
3	0.216506		18	0.201969	0.203002
4	0.207360	0.208112	19	0.202112	0.202916
5	0.209631	0.206919	20	0.201743	0.202838
6	0.205586	0.206077	21	0.201859	0.202767
7	0.206699	0.205453	22	0.201554	0.202702
8	0.204419	0.204973	23	0.201650	0.202643
9	0.205078	0.204591	24	0.201394	0.202589
10	0.203614	0.204282	25	0.201475	0.202539
11	0.204050	0.204026	26	0.201256	0.202492
12	0.203031	0.203810	27	0.201326	0.202450
13	0.203340	0.203626	28	0.201137	0.202410
14	0.202590	0.203467	29	0.201198	0.202372
15	0.202821	0.203328	30	0.201033	0.202338

Case $n = 9$. Similarly, we may restrict x to $[4/9, 1/2]$, and on this interval

$$\begin{aligned} c_9(x) &= 210x^5(1-x)^5 \\ &\quad + 3\{(1-x)^9(x-\tfrac{1}{3}) + (1-x)^8\, x(9x-4) \\ &\quad + x^8(1-x)(5-9x) + x^9(\tfrac{2}{3}-x)\}. \end{aligned}$$

This expression is maximal at $x = \frac{1}{2}$ with $c_9(\frac{1}{2}) = 109/512 = 0.212891 < c_5$.

Case $n = 11$. Restricting x to $[0.49, 0.50]$ we improve slightly on the inequalities (6.1) and (6.2). As $Q_{11}(0) < P_{11}(0) - 0.17$ and $Q_{11}(1) < P_{11}(1) - 0.20$, it follows that the estimate (6.2) can be improved by

$$11^{1/2}\{0.17(1-x)^{11} + 0.20x^{11}\} > 0.000550 \qquad (x \in [0.49, 0.50]).$$

From Table I it follows that this suffices to prove that $c_{11} < c_5$. This concludes the proof of Theorem 6.1.

Remark 6.1. From the proof of Theorem 6.1 it does not follow that $c_n = c_n(\frac{1}{2})$ for $n = 7$, 9, and 11. Careful computation however, shows that this is true.

7. The Limiting Behavior of $c_n(x)$ and c_n

We shall prove

THEOREM 7.1. *For $c_n(x)$ and c_n as defined in* (1.6) *and* (1.3) (*cf.* (1.7)), *we have*

$$c(x) := \lim_{n\to\infty} c_n(x) = \left(\frac{X}{2\pi}\right)^{1/2} + 2X^{1/2} \sum_{j=1}^{\infty} \int_{jX^{-1/2}}^{\infty} (u - jX^{-1/2})\, \varphi(u)\, du \qquad (0 < x < 1), \quad (7.1)$$

$$\lim_{n\to\infty} c_n = c(\tfrac{1}{2}) = (2\pi)^{-1/2} \left\{\tfrac{1}{2} + \sum_{j=1}^{\infty} e^{-2j^2}\right\} - 2\sum_{j=1}^{\infty} j(1 - \Phi(2j)) = 0.20796899. \quad (7.2)$$

Here $X = x(1 - x)$, $\varphi(x) = (2\pi)^{-1/2} \exp(-\frac{1}{2}x^2)$, *and* $\Phi(x) = \int_{-\infty}^{x} \varphi(u)\, du$.

To establish this theorem we state two lemmas; for the proof of Lemma 7.2 we refer to [7].

LEMMA 7.1. *If U is a nonnegative random variable with distribution function F, then, denoting expectation by E,*

$$E(U - a)_+ = \int_a^{\infty} (1 - F(u))\, du \qquad (a \geqslant 0). \quad (7.3)$$

LEMMA 7.2. *If V_n is a binomial random variable with expectation nx and variance nX, and if we put $U_n = (V_n - nx)(nX)^{-1/2}$, then for the distribution function F_n of $|U_n|$ one has*

$$1 - F_n(u) \leqslant 2\exp(-u^2 x(1 - x)) \qquad (u \geqslant 0;\ 0 < x < 1).$$

Proof of Theorem 7.1. Using Lemmas 7.1 and 7.2, in view of (4.1) we have

$$c_n(x) = X^{1/2} \left\{\tfrac{1}{2}E\,|U_n| + \sum_{j=1}^{\infty} E(|U_n| - jX^{-1/2})_+\right\}$$

$$= X^{1/2} \left\{\tfrac{1}{2}\int_0^{\infty} (1 - F_n(u))\, du + \sum_{j=1}^{\infty} \int_{jX^{-1/2}}^{\infty} (1 - F_n(u))\, du\right\}. \quad (7.4)$$

By the Berry–Esseen version of the central limit theorem [3, p. 542], $1 - F_n(u)$ tends to $2(1 - \Phi(u))$, uniformly in $x \in [\delta, 1 - \delta]$ for any $\delta > 0$. By Lemma 7.2 the integrals in (7.4) converge uniformly in j, n and $x \in [\delta, 1 - \delta]$, and the sum converges uniformly in n and x. It then follows that, uniformly in $x \in [\delta, 1-\delta]$,

$$\lim_{n\to\infty} c_n(x) = X^{1/2} \left\{ \int_0^\infty (1-\Phi(u))\,du + 2\sum_{j=1}^{\infty} \int_{jX^{-1/2}}^{\infty} (1-\Phi(u))\,du \right\}, \tag{7.5}$$

which by (7.3) is equivalent to (7.1). We note that x_n satisfying $c_n = c_n(x_n)$ is bounded away from 0 and 1 (cf. (5.6)). Now using the fact that (7.5) holds uniformly in $x \in [\delta, 1-\delta]$, we obtain (cf. (1.5))

$$c := \lim_{n\to\infty} c_n = \lim_{n\to\infty} c_n(\tfrac{1}{2}) = \tfrac{1}{2}\int_0^\infty (1-\Phi(u))\,du + 2\sum_{j=1}^{\infty} \int_{2j}^{\infty} (1-\Phi(u))\,du,$$

which is equivalent to (7.2). The numerical value can be obtained from [1, pp. 968–972].

Concluding Remarks

The techniques used in this paper can be employed to treat similar problems for other values of δ in $\omega_1(f; \delta)$. For $\delta = n^{-1}$ this has been done in [8]. The value $\delta = n^{-1/2}$ seems to be the most natural, whereas $\delta = n^{-1}$ yields the most explicit results.

Estimates for different values of δ can be connected by the obvious inequality $\omega_1(\cdot; \delta_2) \leqslant (\delta_2/\delta_1 + 1)\, \omega_1(\cdot; \delta_1)$ for $\delta_2 > \delta_1$. This has been done in [4], where local results (i.e., results containing x) for $\delta = n^{-1}$ and $\delta = n^{-1/2}$ are derived, which are weaker than the results obtained in [8] and the present paper.

It may be possible to improve somewhat on the results or the proofs in [2, 9, 10] by the type of argument used in this paper.

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